LETTER TO THE EDITOR

Period-doubling route to chaos for a global variable of a probabilistic automata network

N Boccara†‡ and M Roger†

† DRECAM/SPEC, CE-Saclay, 91191 Gif-sur-Yvette Cedex, France
‡ Department of Physics, University of Illinois, Chicago, IL 60680, USA

Abstract. As a function of a parameter characterizing the degree of mixing of site values, the density of non-zero sites of some one-dimensional cellular automata is shown to exhibit a sequence of period-doubling bifurcations and to behave chaotically when the degree of mixing is sufficiently large. The automata network rules which are considered appear to be useful to model complex systems, as in epidemiology, in which the motion of the individuals is believed to play an important role.

In a recent paper Boccara and Cheong (1992) have presented and studied automata network epidemic models. In these models, the rule for the spread of the infectious disease consists of two subrules. The first one, applied synchronously, models infection and removal (or recovery), and the second, applied sequentially, describes the motion of the individuals. The spatial correlations created by the application of the first subrule are partially destroyed according to the degree of mixing of the population which follows from the application of the second subrule.

Since automata networks of this type could be used to model other complex systems in which motion is believed to play an important role as, for instance, interacting populations in ecology, we have studied some general properties of automata networks whose rules consist of two subrules defined as follows. The first one is a standard two-state $(0, 1)$ cellular-automaton rule, which may be deterministic or probabilistic, and the second moves a fraction of non-zero sites. More precisely, the parameter m that measures the degree of mixing is defined as follows. A site whose value is one is selected at random and swapped with another site value (either zero or one) selected at random. This operation is repeated $mc(m, t)N$ times, where N is the total number of sites and $c(m, t)$ the density of non-zero sites at time t. It is important to note that this process does not change the value of $c(m, t)$. With this definition, the number of effective moves of non-zero sites is $mc(m, t)(1 - c(m, t))N$.

In some cases, it might be of interest to define the range of the move, i.e., the maximum distance between the sites whose values are swapped. The greater the range, the more effective the mixing process.

When m is large, all the correlations created by the cellular-automaton rule are destroyed, and the evolution of the density of non-zero sites is correctly predicted by a mean-field type approximation. This approximation is often far from being correct (Bidaux *et al* 1989, Chaté and Manneville 1991, 1992). In this letter, we study some

one-dimensional cellular automata whose density of non-zero sites $c(0,t)$ tends, for $m = 0$, to a fixed point $c(0,\infty)$ as t tends to ∞, whereas their mean-field maps are chaotic. Therefore, varying the control parameter m, which route to chaos is going to follow the stationary value of the density of non-zero sites?

We restrict our study to 'game of life'-type totalistic cellular-automaton rules. The state at time t of a one-dimensional cellular automaton being represented by a function $i \mapsto s(t,i)$ from the set of integers $\mathbb{Z}$ to the set of states $\{0,1\}$, the equation

$$s(t+1,i) = f\big(s(t,i-r) + s(t,i-r+1) + \cdots + s(t,i+r)\big)$$

where f is a map from $\{0,1,\ldots,2r+1\}$ to $\{0,1\}$, determines the evolution of a deterministic range-r totalistic cellular automaton.

f is such that

$$f(x) = \begin{cases} 1 & \text{if } S_{\min} \leqslant x \leqslant S_{\max} \\ 0 & \text{otherwise.} \end{cases}$$

If a neighbourhood is too poorly populated or too crowded, the central site 'dies', otherwise it 'survives'.

$S_{\min}$, $S_{\max}$ and r are chosen to obtain a chaotic mean-field map (i.e. r should be larger than 1, $S_{\min}$ not too large and $S_{\max}$ not too small). In order to have well defined global variables, like $c(0,\infty)$, the automaton rule should be of class III in the sense of Wolfram (1983). Many rules with r ranging from 2 to 7 have been investigated and the same qualitative features have been obtained. Typical bifurcation diagrams for two range-3 rules are given in figures 1 and 2. The simulations have been done on a chain of $N = 10^6$ sites with periodic boundary conditions. Figure 1 represents the bifurcation diagram of $c(m,\infty)$ for $r = 3$, $S_{\min} = 2$ and $S_{\max} = 6$ (rule 124 according to Wolfram's notation) and figure 2(a) the bifurcation diagram for $r = 3$, $S_{\min} = 1$ and $S_{\max} = 6$ (rule 126). The route to chaos followed by $c(m,\infty)$ is the familiar cascade of period-doubling bifurcations.
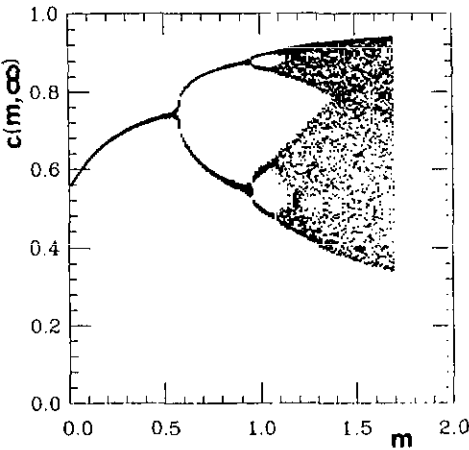

Figure 1. Bifurcation diagram for the stationary value $c(m,\infty)$ of the density of non-zero sites for range-3 totalistic rule 124.

The main difference in these diagrams occurs at low m. $\partial c(m,\infty)/\partial m$ is finite at the origin for rule 124 (figure 1) whereas this derivative is infinite for rule 126
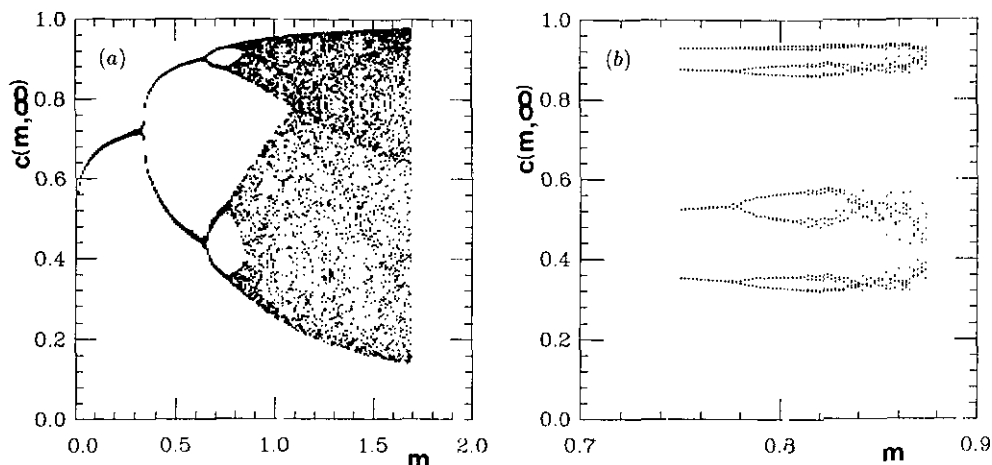
Figure 2. Bifurcation diagram for the stationary value $c(m, \infty)$ of the density of non-zero sites for range-3 totalistic rule 126. (a) Overall diagram with $N = 10^6$ sites. (b) Blow-up of the bifurcation diagram with $N = 16 \times 10^6$ sites. To better see the period-16 cycle, the points represent averages over five cycles.

(figure 2(a)). The asymptotic behaviour for small m of $c(m, \infty)$ is related to the asymptotic behaviour of $c(0, t)$ for large t (Boccara and Roger 1992). The first case is observed when $c(0, t)$ tends exponentially to $c(0, \infty)$ while the second is a consequence of a power-law behaviour of $c(0, t)$. This latter behaviour is characteristic of automata whose evolution toward their limit set may be viewed, after a short transient, as an assembly of interacting defects in a regular background. As a result of the interactions the number of defects decreases as a power of time and, therefore, we have $c(0, t) - c(0, \infty) \sim t^{-\gamma}$ (Boccara *et al* 1991), where $0 < \gamma < 1$. Rule 126 is indeed a block transform (Boccara and Roger 1990) of range-1 totalistic rule 6. Its limit set consists of sequences of zeros and ones whose lengths are multiples of 6. The distribution of the sequences of zeros is identical to the distribution of the sequences of ones, which implies that the asymptotic density of non-zero sites is $\frac{1}{2}$, and the average number of sequences of length $6n$ per site is $1/3 \times 2^{n+3}$. The convergence to the limit set is obtained through the elimination of defects d_n corresponding to sequences of zeros or ones whose lengths are $n \pmod 6$. These defects combine according to the simple law: $d_m + d_n \rightarrow d_q$ with $m + n = q$ $\pmod 6$, they experience a diffusive motion and their number decreases as $t^{-1/2}$.

We shall now focus our attention on this second rule. Since the mean-field approximation reads

$$c(t+1) = 1 - c(t)^7 - (1 - c(t))^7 = 7c(t)(1 - c(t))(1 - 2c(t) + 3c(t)^2 - 2c(t)^3 + c(t)^4)$$

its mean field map has a quadratic maximum. The universality class corresponding to such a map has been studied extensively (Feigenbaum 1978, Bergé *et al* 1984) and it is interesting to investigate more carefully the bifurcation diagram represented in figure 2(a). To obtain a higher precision on the location of the first bifurcations, part of this diagram has been investigated with a chain of a larger length ($N = 16 \times 10^6$) and smaller steps in m. We have also performed averages over five cycles to better see the period-16 cycle. The results are represented in figure 2(b). The first four bifurcations (figures 2) are located at $m_1 = 0.340 \pm 0.005$, $m_2 = 0.655 \pm 0.005$,

$m_3 = 0{,}780 \pm 0.001$, and $m_4 = 0.813 \pm 0.001$. Due to the fact that the size of the one-dimensional lattice is finite, the full sequence of period-doubling bifurcations cannot be observed. The Feigenbaum constant δ (Feigenbaum 1978) is the limit as n goes to ∞ of the sequence (δ_n), where $\delta_n = (m_{n+1} - m_n)/(m_{n+2} - m_{n+1})$. From the first four bifurcation points, we have $\delta_1 = 2.5$ and $\delta_2 = 3.8$ within 10%. δ may also be estimated from the inverse cascade. We have found $\overline{m}_1 = 1.095 \pm 0.005$, $\overline{m}_2 = 0.859 \pm 0.001$, and $\overline{m}_3 = 0.827 \pm 0.001$, which gives $\overline{\delta}_1 = 7.4$ within 10%. These results are similar to those obtained from the iterations of a unimodal map, and if we could have observed more bifurcations, δ_n would have probably have come closer to the universal value 4.67.

In a periodic attractor, the order in which its constituent points are visited is important. With the usual notation (Bergé *et al* 1984), we obtain for period four the sequence of points 2-0-3-1, for period eight: 2-6-0-4-3-7-5-1, for period sixteen: 2-10-14-6-8-0-12-4-3-11-15-7-5-13-9-1, in agreement with the universal sequences (Bergé *et al* 1984).

At first sight, our result might appear rather surprising. Why this route to chaos? Since the existence of a map in this context is not evident, why should we observe the universal behaviour corresponding to maps with a quadratic maximum?
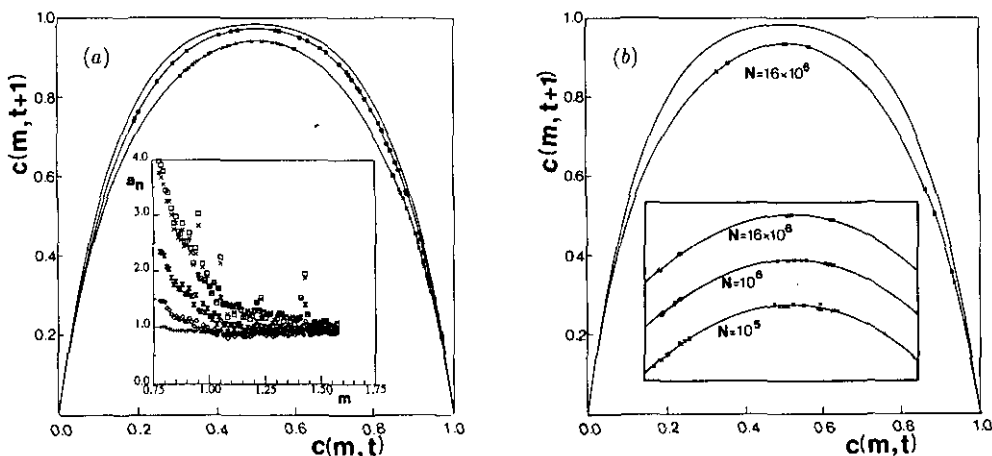

Figure 3. $(c(t+1), c(t))$ plot for different values of t and m (rule 126). (*a*) Two typical fits with $m = 0.85$ (crosses) and $m = 1.40$ (squares). The upper curve represents the mean-field map. The inset shows the variations, as a function of m, of the coefficients of the polynomial

$$f(x) = 7x(1 - x)[a_0(m) - 2a_1(m)x + 3a_2(m)x^2 - 2a_3(m)x^3 + a_4(m)x^4]$$

$a_0(m)$: (+), $a_1(m)$: (◇), $a_2(m)$: (✕), $a_3(m)$: (□), $a_4(m)$: (×). (*b*) Influence of the number of sites N for a typical fit ($m = 0.80$) in a period 2^3 regime. The two upper curves represent the mean-field map and the polynomial fit for $N = 16 \times 10^6$. The inset shows a blow-up of the central part with $N = 10^5$, $N = 10^6$ and $N = 16 \times 10^6$.

It is interesting to plot in the periodic and in the chaotic regime $c(m, t + 1)$ as a function of $c(m, t)$ for given values of m. Figure 3(*a*) represents typical plots with $m = 0.85$ (crosses) and $m = 1.40$ (open squares) for a 10^6-site chain. For each value of m we used a least-squares approximation to determine a polynomial of the form $f_m(x) = 7x(1 - x)P_m(x)$. The degree of the polynomial $P_m(x)$ is chosen in

the following way: successive approximations with increasing degree are determined, and we stop when increasing the degree does not improve the root mean square residual σ. We have found that degree equal to 4 for $P(x)$ was most relevant in the range $0.75 < m < 1.7$ which has been investigated (σ decreases by a factor of order 3 from degree 3 to 4 and is practically constant beyond degree 4). Since $f(x)$ has the same degree as the mean field map it is interesting to write

$$P(x) = a_0(m) - 2a_1(m)x + 3a_2(m)x^2 - 2a_3(m)x^3 + a_4(m)x^4.$$

The mean field map is recovered for $a_0 = a_1 = a_2 = a_3 = a_4 = 1$. In this whole range, σ is roughly constant and quite small (10^{-3}), which seems to indicate the existence of a map $f_m(x)$ for every m. The inset shows the variations of the coefficients of $P(x)$ as a function of m. Within the accuracy of our fit, those coefficients satisfy the relation: $a_4 = a_3 = 3a_2 - 2a_1$, which corresponds to the symmetry of the map with respect to the vertical axis $x = \frac{1}{2}$. The evolution of the map as a function of m is far from a simple scaling of the mean-field map with a factor $\lambda(m)$. While $a_0(m)$ is roughly constant, the coefficients of higher degree increase drastically when m decreases. Some points at $m = 0.95$, $m = 1.05$, $m = 1.23$, $m = 1.43$ are clearly out of the error bars. The first and last values of m correspond to narrow windows of periodicity, that—although hardly visible at this scale—can be seen in figure 2(b). The first one has period six with a sequence (according to the previous notation) 2-0-4-3-5-1, in agreement with the universal behaviour for the first window of period six (Bergé *et al* 1984). The last one has period five: 2-0-3-4-1. Further simulations with a larger number of sites and smaller steps in m are necessary to identify other narrower windows.

An important question now is: does the root mean square residual σ vanish as the number of sites N goes to infinity or is there some intrinsic noise as far as we have not reached the mean field limit? In the latter case, only a truncated sequence of bifurcations could be observed even in the limit $N \to \infty$ (Crutchfield *et al* 1981, Shraiman *et al* 1981). Figure 3(b) represents the polynomial fit corresponding to $m = 0.80$ (period 2^3 regime) for different values of N. The upper curve shows the global fit with $N = 16 \times 10^6$. The inset represents a blow-up of the central part for various N. The mean square root residual σ of the polynomial fit is respectively 2×10^{-3}, 10^{-3} and 0.75×10^{-3} for $N = 10^5$, $N = 10^6$ and $N = 16 \times 10^6$. We do not know whether this small decrease of σ is significant, but it is really much slower than $1/\sqrt{N}$. In contrast, the spreading of the points along the map decreases roughly as $1/\sqrt{N}$ to converge towards a sequence of eight points.

Although it is difficult to conclude definitively, the small decrease of σ as N increases from 10^5 to 1.6×10^7, and the fact that when N is fixed, σ does not decrease significantly when we approach the mean-field limit seems to indicate the existence of a map without noise as $N \to \infty$.

We have restricted our study to one-dimensional systems. The same behaviour is expected in two dimensions for similar class-3 'game of life'-type automata. The results shown here are quite general and could be useful in modelling a variety of biological systems where the motion of individuals is important.

Recently Chaté and Manneville (1991, 1992) and Gallas *et al* (1991) have observed collective behaviours in a class of four- and five-dimensional cellular automata. Despite the rather high space dimensionality of these automata, their stationary densities of non-zero sites do not behave as predicted by the corresponding mean-field maps

as in Bidaux *et al* (1989). These automata exhibit, however, unexpected periodic or quasiperiodic behaviours, showing that class-3 cellular automata may have a more diverse behaviour than suggested by Wolfram (1983).

We are indebted to P Manneville and H Chaté for many useful suggestions and exciting discussions.

References

Bergé P, Pomeau Y and Vidal C 1984 *Order Within Chaos, Towards a Deterministic Approach to Turbulence* (Paris: Hermann)
Bidaux R, Boccara N and Chaté H 1989 *Phys. Rev.* A **39** 3094–105
Boccara N and Cheong K 1992 *J. Phys. A: Math. Gen.* **25** 2447–61
Boccara N, Nasser J and Roger M 1991 *Phys. Rev.* A **44** 866–75
Boccara N and Roger M 1990 *J. Phys. A: Math. Gen.* **24** 1849–65
—— 1992 to appear
Chaté H and Manneville P 1991 *Europhys. Lett.* **14** 409–13
—— 1992 *Prog. Theor. Phys.* **87** 1–60
Crutchfield J, Nauenberg M and Rudnick J 1981 *Phys. Rev. Lett.* **46** 933–5
Feigenbaum M J 1978 *J. Stat. Phys.* **19** 25–52
Gallas J A C, Grassberger P, Hermann H J and Ueberholz P 1991 *Physica* **180A** 19–41
Shraiman B, Wayne C E and Martin P C 1981 *Phys. Rev. Lett.* **46** 935–9
Wolfram S 1983 *Rev. Mod. Phys.* **55** 601–44